PHYS 798C Spring 2024
Lecture 18 Summary

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I. GINZBURG-LANDAU THEORY CONTINUED - THE STRUCTURE OF INDIVIDUAL VORTICES

II. ESTIMATION OF H_{c1}

How do vortices first enter a type-II superconductor? As the applied field is increased from 0, the superconductor will maintain the Meissner state by creating screening currents to keep the flux out. It costs energy to create such screening currents. On the other hand, allowing the field to penetrate will force the superconductor to give up some condensation energy in the core of the vortex where the order parameter is suppressed (as seen in the $\psi(x,y)$ profiles in the Abrikosov vortex lattice). The first vortex will enter when these two energies are comparable. In other words when
$\frac{\mu_0 H_{c1}^2}{2}\pi\lambda_{eff}^2 L \approx \frac{\mu_0 H_c^2}{2}\pi\xi_{GL}^2 L$, where L is the length of the vortex in the superconductor. The left hand side is an estimate of the energy required to exclude a magnetic field of magntidue H_{c1} in a 'tube' of radius λ_{eff}, while the right hand side is the condensation energy lost when the vortex core goes normal. The equality yields,
$H_c \approx \kappa H_{c1}$ with $\kappa = \lambda_{eff}/\xi_{GL}$. Hence $H_{c1} < H_c$ in type-II superconductors. Using our earlier result for the upper critical field, $H_{c2} = \sqrt{2}\kappa H_c$, one can find
$H_c = \sqrt{H_{c1}H_{c2}/\sqrt{2}}$, essentially the geometric mean.

III. STRUCTURE OF AN ISOLATED VORTEX

We will attempt to solve the full nonlinear GL equation which is coupled to the current equation, self consistently. We make a number of assumptions based on the vortex lattice solution:
1. The magnetic field is in the z direction, $\vec{H} = H\hat{z}$. We shall ignore any variation of the solution in the z direction (i.e. $\partial/\partial z$, etc. will be ignored).
2. The magnetic field is supported by currents that flow in the $x-y$ plane. Hence both $\vec{J}$ and $\vec{A}$ will be confined to that plane.
3. The solution will have full cylindrical symmetry. We will use cylindrical coordinates in the form of (r,θ,z). The order parameter will be of the form $\psi(\vec{r}) = \psi_\infty f(r)e^{i\theta}$. The explicit θ dependence is intentional. It gives rise to a phase pickup of 2π upon moving the wavefunction in a closed loop around the vortex core in a plane perpendicular to the magnetic field direction. This corresponds to the vortex carrying $+1$ unit of magnetic flux. Using a solution of the form $\psi(\vec{r}) \propto e^{-i\theta}$ creates a vortex with -1 flux quantum, and is called an anti-vortex. We shall see that when a vortex meets an anti-vortex, they annihilate each other! Note that zeros of a complex scalar function are endowed with a circulation, as explained in the paper by Neu posted on the class website.
4. With this cylindrical symmetry, the vector potential is constrained to have only a radial dependence and a $\hat{\theta}$ direction: $\vec{A}(\vec{r}) = A(r)\hat{\theta}$.

With these observations, the coupled GL and current equations become:
$f - f^3 - \xi_{GL}^2\left[\left(\frac{1}{r} - \frac{2\pi}{\Phi_0}A(r)\right)^2 f - \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)\right] = 0$, where $f = \psi/\psi_\infty$ and the current density expression is
$\vec{J} = \frac{e^*}{m^*}\psi_\infty^2 f^2(r)\left[\frac{\hbar}{r}\hat{\theta} - e^*\vec{A}(r)\hat{\theta}\right]$ with
$\vec{B} = \vec{\nabla}\times\vec{A} = \hat{z}\frac{1}{r}\frac{\partial}{\partial r}\left(rA(r)\right)$.

One can solve for the vector potential in terms of the magnetic field by integrating the last expression, $A(r) = \frac{1}{r}\int_0^r \mu_0 h(r')r'dr'$, where $h(r)$ is the microscopic magnetic field.

Now examine the vector potential at small r (near the vortex core) and at large r, in turn:
1. As $r \to 0$, on the scale of the GL coherence length, we expect that $h(r)$ is uniform since it is screened

out on the much larger length scale λ_{eff} in the type-II high-κ limit. Hence $h(r) \approx h(0)$ and we have, $A(r \to 0) = \frac{\mu_0 h(0)}{2} r$.
2. As $r \to \infty$ the integral for $A(r)$ encompasses all of the flux in the vortex, which we know is Φ_0, and the result is $A(r \to \infty) = \frac{\Phi_0}{2\pi r}$.

Now examine the GL equation in each of these limits:
1. As $r \to 0$ we use the solution for $A(r)$ given above to find,
$f - f^3 - \xi_{GL}^2 \left[\left(\frac{1}{r} - \frac{2\pi}{\Phi_0} \frac{\mu_0 h(0)}{2} r \right)^2 f - \frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \right] = 0.$
We try a power-law solution good near the origin, $f(r) = cr^n$. One finds that to leading order $n = 1$, which is exactly what we saw in the solution for $\psi(x, y)$ in the vortex lattice solution - ice cream cones. To next order one finds a correction to $f \propto r$ of cubic order in r/ξ_{GL}, with a minus sign. The solution can therefore be approximated as $f(r) \approx \tanh(r/\xi_{GL})$. This has the linear behavior at small r and the asymptotic behavior $f = 1$ far from the vortex core, as expected. Hence we now have a solution that also works at large r.

Going back to the vector potential and magnetic field, we can utilize the fact that screening occurs on longer length scales than the core diameter ($\kappa >> 1$), hence $f(r) \approx 1$ for $r > \xi_{GL}$ in the current density equation above. With this, we can re-write the current density equation as,
$\mu_0 \lambda_{eff}^2 \vec{J} + A(r)\widehat{\theta} = \frac{\Phi_0}{2\pi r}\widehat{\theta}$
The left-hand side is what we formerly called the generalized London relation (see lecture 4). Taking the time derivative gives the first London equation, while taking the curl gives the second London equation. In this case we have an inhomogeneous equation with a source term on the right hand side.

Taking the curl of both sides gives a second London equation with a vorticity source term:
$\mu_0 \lambda_{eff}^2 \vec{\nabla} \times \vec{J} + \vec{\nabla} \times (A(r)\widehat{\theta}) = \vec{\nabla} \times \left(\frac{\Phi_0}{2\pi r}\widehat{\theta} \right)$
The right hand side evaluates to a delta function at the origin: $\vec{\nabla} \times \left(\frac{\Phi_0}{2\pi r}\widehat{\theta} \right) = \Phi_0 \delta_2(r)\widehat{z}$, which we call the vorticity $\vec{V}(\vec{r})$. We can write the resulting equation as,
$\mu_0 \lambda_{eff}^2 \vec{\nabla} \times \vec{J} + \mu_0 \vec{h}(r) = \vec{V}(\vec{r})$.
To proceed with this 'mixed' equation, use the Maxwell equation $\vec{\nabla} \times \vec{h} = \vec{J}$ to get,
$\nabla^2 \vec{h} - \frac{1}{\lambda_{eff}^2} \vec{h} = -\frac{\Phi_0}{\mu_0 \lambda_{eff}^2} \delta_2(r)\widehat{z}$. This (linear) equation has an exact solution!

IV. FIELDS AND CURRENTS OF AN ISOLATED VORTEX

This equation for the microscopic magnetic field of an isolated vortex has an exact solution, the zeroth order Hankel function of complex argument $K_0(x)$,
$\vec{h}(r) = \frac{\Phi_0}{2\pi \mu_0 \lambda_{eff}^2} K_0(r/\lambda_{eff})\widehat{z}$.

This solution has the following asymptotic forms,
For large r, $r > \lambda_{eff}$ it goes as $h(r) \propto r^{-1/2} e^{-r/\lambda_{eff}}$. We know that far from a planar surface where a magnetic field is applied, we expect that $h(r) \propto e^{-r/\lambda_{eff}}$. The difference here is that the magnetic field is created by a line-source, rather than a planar source.
And for $\xi_{GL} < r < \lambda_{eff}$ the field goes as $h(r) \propto \left[\log\left(\frac{\lambda_{eff}}{r} \right) + 0.12 \right]$. This logarithmic dependence has important consequences for vortex-vortex interactions in the Kosterlitz-Thouless phase transition for a 2D superconductor, discussed in Lecture 27.

The class web site shows the full solution, as well as these asymptotic forms, for the field profile as a function of radius. Once you are several times λ_{eff} away from the vortex core, there is essentially no field visible.

The currents can be calculated from the relation $\vec{\nabla} \times \vec{h} = \vec{J}$, and the result is,
$\vec{J}(r) = \frac{\Phi_0}{2\pi \mu_0 \lambda_{eff}^3} K_1(r/\lambda_{eff})\widehat{\theta}$.
The current distribution has the following asymptotic forms,
For large r, $r > \lambda_{eff}$ it goes as $J(r) \propto r^{-1/2} e^{-r/\lambda_{eff}}$,
and for $\xi_{GL} < r < \lambda_{eff}$ it goes as $J(r) \propto 1/r$. This can be re-written as,

$\vec{J} = |\psi|^2 e^* v_{s\theta} \widehat{\theta}$, with $v_{s\theta} = \frac{\hbar}{m^* r}$. This result has an interesting interpretation. It can be written as $m^* v_{s\theta} r = \hbar$. In other words it resembles the Bohr-Sommerfeld quantization rule that $\oint pdq = n\hbar$, where in this case $n = 1$. In other words the supercurrent around a single vortex carries one unit of angular momentum.

The apparent divergence of the current at small r is cut off by the fact that $v_{s\theta}$ will eventually exceed the critical velocity v_c and the order parameter will be suppressed in the core. One can show that the kinetic energy density in the current flow is comparable to the condensation energy density at the edge of the core $r \sim \xi_{GL}$.